YITP-SB-00-71

Colours associated to non simply-laced Lie algebras and exact S-matrices

Christian Korff

C.N. Yang Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, N.Y. 11794-3840

Abstract

A new set of exact scattering matrices in 1+1 dimensions is proposed by solving the bootstrap equations. Extending earlier constructions of colour valued scattering matrices this new set has its colour structure associated to non simply-laced Lie algebras. This in particular leads to a coupling of different affine Toda models whose fusing structure has to be matched in a suitable manner. The definition of the new S-matrices is motivated by the semi-classical particle spectrum of the non simply-laced Homogeneous Sine-Gordon (HSG) models, which are integrable perturbations of WZNW cosets. In particular, the S-matrices of the simply-laced HSG models are recovered as a special case.

PACS numbers: 11.10Kk, 11.55.Ds

korff@insti.physics.sunysb.edu

1 Introduction

In the context of 1+1 dimensional integrable quantum field theories the idea of analytic continuation allows for the explicit construction of exact scattering amplitudes by means of the bootstrap approach [1]. In integrable field theories each scattering process can be decomposed into two-particle processes reducing the problem of calculating the S-matrix to determining the two-particle amplitude $S_{AB}(\theta)$, where A, B label the particle types and θ is the rapidity variable. The analytic continuation of S_{AB} is subject to the following set of functional equations,

$$S_{AB}(\theta)S_{BA}(-\theta) = 1 \tag{1}$$

$$S_{AB}(\theta - i\pi)S_{A\bar{B}}(\theta) = 1 \tag{2}$$

$$S_{DA}(\theta)S_{DB}(\theta + iu_{AB}^{C})S_{DC}(\theta + iu_{AB}^{C} + iu_{BC}^{A}) = 1\,. \tag{3}$$

Equations (1) and (2) reflect the physical constraints of unitarity and crossing symmetry, respectively. Equation (3) is the bootstrap requirement associated with the fusing process $A + B \to \bar{C}$ and plays the most important role in the construction*, since it incorporates the bound state structure of the integrable quantum field theory via the fusing angles u_{AB}^{C}. Which fusing processes occur is usually inferred from the classical Lagrangian of the theory either by means of perturbation theory or a semi-classical analysis. In this manner the solution of the bootstrap equations is linked to a specific field theory identified by a concrete Lagrangian.

However, one might look for solutions of the functional equations independent whether or not a classical Lagrangian formulation of a field theory is given. Indeed, one of the remarkable messages of the bootstrap approach is that every consistent solution to equations (1)-(3) can be interpreted as a two-particle scattering amplitude defining implicitly an integrable quantum field theory. This point of view motivated the construction of a whole set of new factorizable scattering matrices with colour values in [3] by extending S-matrices proposed earlier [4] in the context of the so-called simply-laced Homogeneous Sine-Gordon (HSG) models [5] to a much larger class. In the present work the construction scheme outlined in [3] will be generalized to involve also non simply-laced Lie algebras for the colour values by choosing the semi-classical particle spectrum [6] of the so-called *non simply-laced* HSG models as input data for the bootstrap (3). The resulting factorizable S-matrices might be interpreted as possible candidates for these integrable quantum field theories. As a preparatory step for the construction it is briefly recalled how the Lie algebraic structures enter in the S-matrix construction.

The Lie algebraic methods used in [3] to provide consistent solutions to the bootstrap equations originate in affine Toda field theory [7]. The key feature exploited is the splitting of the ADE affine Toda S-matrix [8] into a minimal part $\mathcal{S}^{g'}$ containing all the physical poles and a CDD-factor [9] $\mathcal{F}^{g'}$ displaying the coupling dependence. Here the simple Lie algebra $g' = ADE$ fixes the affine Toda model in question. As explained in [3]

*The Yang-Baxter [2], which in general also arises in this context, will be trivially fulfilled, since we are going to consider only theories where the S-matrices are assumed to be diagonal.

this model can be 'multiplied' by labeling particles through two quantum numbers $A = (a,i)$ each corresponding to a vertex in the Dynkin diagrams of a *pair* of simple *simply-laced* Lie algebras $g'|g$. As already pointed out the first algebra g' fixes the scattering amplitude from affine Toda field theory (ATFT), while the second yields the so-called colour degree of freedom governing the interaction type. Particles of the same colour i are chosen to interact via the minimal affine Toda S-matrix $S_{ab}^{ii}(\theta) := \mathcal{S}_{ab}^{g'}(\theta)$, while particles of different colours i,j scatter via the CDD-factor provided the corresponding vertices in the Dynkin diagram of g are linked to each other,

$$S_{ab}^{ij}(\theta) := \left\{ \begin{array}{cc} 1\,, & A_{ij} = 0 \\ \eta_{ab}^{ij} \mathcal{F}_{ab}^{g'}(\theta, B=1)^{\frac{1}{2}} \neq 1\,, & A_{ij} < 0 \end{array} \right. . \tag{4}$$

Here A is the Cartan matrix associated with the colour Lie algebra g and the square root has been taken of the CDD-factor $\mathcal{F}^{g'}$ at effective coupling $B = 1$. The factor $\eta_{ab}^{ij} = \bar{\eta}_{ba}^{ji}$ in front is a constant of modulus one crucial for satisfying the bootstrap requirements and leading to parity violation, $S_{ab}^{ij}(\theta) \neq S_{ba}^{ji}(\theta)$ (see [4, 3] for details). Note that in order to avoid 'bulky' expressions the notation used in [3] has been adopted denoting the scattering amplitude S_{AB} with $A = (a,i)$, $B = (b,j)$ by S_{ab}^{ij}.

Besides yielding numerous new solutions it was shown that the above construction (4) also recovers several known cases as specific subsets, one being the scaling or minimal affine Toda models [10, 8] when choosing $g'|su(2)$ and the other the already mentioned class of simply-laced HSG theories when selecting $su(k)|g$. The common feature of these theories is their interpretation as integrable perturbations of parafermionic conformal field theories [11, 12]. The latter are obtained by the GKO coset construction [13] from WZNW models [14]. For example the minimal affine Toda or scaling models are related to cosets of the form $g'_1 \oplus g'_1/g'_2$, while the HSG theories are linked to $g_k/u(1)^{\times \operatorname{rank} g}$. Here the lower index refers to the so called *level* labeling the representation of the affine extension of g, g'. This matching between exact S-matrices constructed via the bootstrap approach and conformal field theories can be achieved by several methods.

For instance, the thermodynamic Bethe ansatz (TBA) [15, 16] allows to calculate the effective central charge of the underlying conformal models from the two-particle scattering amplitude. For the cases mentioned this analysis has been carried out in [16] and [17]. The general case of all simply-laced colour valued S-matrices was discussed in [3] and the universal formula found for the effective central charge matches with the one obtained by Dunne et al. [18] when generalizing the discussion of parafermionic CFT's in [12]. An alternative approach which enables one to extract information about the underlying UV conformal model is the form factor program [19]. In [20, 21] it has been shown for a series of simply-laced HSG models that besides the central charge even the conformal dimensions and the local operator content can be extracted.

Looking at the conformal cosets mentioned above it is natural to ask about possible extensions when g', g are chosen to be *non simply-laced* simple Lie algebras, i.e. of $BCFG$ type. The construction of S-matrices involving non simply-laced algebras usually turns out to be more complicated. For instance, selecting g' to be non simply-laced the mentioned separation property of the affine Toda S-matrix is spoiled [22] due to

a coupling dependent flow between two dual algebras (see e.g. [23, 24] and references therein). Hence, the construction scheme of [3] breaks down. Nonetheless, in this work it will be demonstrated how the construction can at least be extended to non simply-laced colour algebras g. It will turn out that the non simply-laced structure of g can be accommodated by coupling different ATFT models to each other, i.e. the choice of g' will vary in dependence on the Lie algebraic data of g.

The article is organized as follows. Section 2 is concerned with constructing the set of new S-matrices with non simply-laced colour structure and starts out with defining the asymptotic spectrum of particles. Motivated by the semi-classical analysis of non simply-laced HSG models [6] the different length of the simple roots of $g = BCFG$ will be taken into account by choosing different particle numbers for each colour value $i = 1, ..., \operatorname{rank} g$. In fact, this construction aims at an extension of the simply-laced HSG S-matrices, whence the algebra g' will be chosen to be $su(k_i)$ where the value of $k_i > 1$ will depend on the colour as explained below. In the next step closed formulas are provided for the new set of S-matrices covering all choices of g including the simply-laced algebras. Using the same techniques as in [3] the new S-matrices are shown to be consistent solutions of the functional equations (1), (2) and (3). To demonstrate the working of the general formulas the case $g = B_2$ or $so(5)$ is presented as an example. Section 3 states the conclusions.

2 Colours from non simply-laced Lie algebras

In the following let g denote a simple Lie algebra, $h^\vee$ its dual Coxeter number and $\{\alpha_i\}_{i=1}^{\operatorname{rank} g}$ a set of simple roots. Normalizing the length of the long roots to be $\alpha_{\text{long}}^2 = 2$ it will turn out to be convenient for the subsequent calculations to define the integers

$$t_i := \frac{2}{\alpha_i^2} \in \{1, 2, 3\} \tag{5}$$

which symmetrize the Cartan matrix A associated to g,

$$A_{ij} t_j = A_{ji} t_i \quad \text{with} \quad A_{ij} = t_i \langle \alpha_i, \alpha_j \rangle \ . \tag{6}$$

Motivated by the semi-classical particle spectrum found in [6] assign to each simple root α_i a tower of stable particles whose mass ratios are determined by

$$M_a^i = m_i \sin \frac{\pi a}{k_i} \ , \qquad i = 1, ..., \operatorname{rank} g, \qquad a = 1, ..., k_i - 1, \tag{7}$$

where $k_i := t_i k$ and $k > 1$ is the so-called *level*. Thus, analogous to the discussion in [3] the stable particles are labelled by a pair (a, i) of quantum numbers and the choice (7) links the structure of the mass spectrum for fixed i to the $su(k_i)$ affine Toda model. The $\operatorname{rank} g$ constants m_i are left undetermined and might be all the same or different[†].

[†]Notice that it is always assumed that the quantum particles are distinguishable by some unspecified quantum charge justifying the ansatz of diagonal S-matrices.

Since the mass spectrum resembles the one from ATFT it also inherits the corresponding fusing structure determined by the fusing angles [8]

$$u_{ab}^{c} = \begin{cases} \frac{\pi}{k_i}(a+b), & a+b+c = k_i \\ 2\pi - \frac{\pi}{k_i}(a+b), & a+b+c = 2k_i \end{cases} . \tag{8}$$

Here it is understood that whenever the above conditions on the particle indices a, b, c are not satisfied the fusing process $a + b \to \bar{c}$ is not present in the theory. As discussed in the introduction the corresponding affine Toda scattering amplitude can be expressed as a product of a so-called minimal and a CDD-factor [8],

$$\text{ATFT:} \qquad \mathcal{S}_{ab}^{su(k_i)}(\theta, B) = \mathcal{S}_{ab}^{su(k_i)}(\theta)\mathcal{F}_{ab}^{su(k_i)}(\theta, B), \tag{9}$$

each of which satisfies the functional equations (1)-(3) separately.

In order to proceed as closely as possible to the former construction the particles for fixed quantum number i are assumed to interact via the minimal scattering matrix of $su(k_i)$ ATFT,

$$S_{ab}^{ii}(\theta) = \mathcal{S}_{ab}^{su(k_i)}(\theta) \ . \tag{10}$$

In contrast to the simply-laced case one has now *different* models of ATFT coupled to each other. Thus, when looking for a CDD-interaction between particles of different colours i, j, which is similar to the one explained in the introduction, the first problem one encounters is that the quantum numbers a, b may now run over different ranges, namely $a = 1, ..., k_i - 1$ and $b = 1, ..., k_j - 1$. This obviously excludes the possibility of taking always the same CDD-factor as in the simply-laced case. Secondly, the CDD-factor has now to comply with both the fusing structure of $su(k_i)$ and $su(k_j)$ ATFT. These two problems can be resolved by choosing the CDD-factor belonging to the affine Toda model associated with the shorter root and by a suitable identification of the particles in the different copies.

Let us assume that $t_i > 1$ and $t_j = 1$, which is the only problematic case coupling two different affine Toda models to each other. Because of its particular simple form the fusing condition of $su(k)$ ATFT [8]

$$su(k): \qquad a + b \to \bar{c} \quad \Longleftrightarrow \quad a + b + c = k \text{ or } 2k \tag{11}$$

can be easily translated into a fusing rule of $su(k_i)$ ATFT by multiplication with the constant t_i and by identifying the particles a and $t_i a$ in the $su(k)$ and $su(k_i)$ theory,

$$su(k_i): \qquad t_i a + t_i b \to t_i \bar{c} \quad \Longleftrightarrow \quad t_i a + t_i b + t_i c = k_i \text{ or } 2k_i \tag{12}$$

Moreover, also the corresponding fusing angles of $su(k)$ and $su(k_i)$ coincide under the above prescription,

$$u_{ab}^{c}\big|_{su(k)} = u_{t_i a, t_i b}^{t_i c}\big|_{su(k_i)} \tag{13}$$

Therefore, the $su(k_i)$ affine Toda CDD-factor $\mathcal{F}^{su(k_i)}$ satisfies the bootstrap equation (3) related to $su(k)$,

$$\mathcal{F}^{su(k_i)}_{d,t_i a}(\theta)\mathcal{F}^{su(k_i)}_{d,t_i b}(\theta+iu^c_{ab})\mathcal{F}^{su(k_i)}_{d,t_i c}(\theta+iu^c_{ab}+iu^a_{bc})=1\ . \tag{14}$$

Thus, when analogously to (4) we set

$$S^{ij}_{ab}(\theta)=\begin{cases} 1\ , & A_{ij}=0 \\ \eta^{ij}_{ab}\mathcal{F}^{su(k_i)}_{a,t_i b}(\theta,B=1)^{\frac{1}{2}}\neq 1\ , & A_{ij}<0 \end{cases} \tag{15}$$

the matrix element $S^{ij}_{ab}(\theta)$ satisfies the necessary bootstrap equations of both algebras $su(k_i)$ and $su(k)$ provided that

$$\eta^{ij}_{ab}=\exp i\pi\varepsilon_{ij}\left(A^{-1}_{su(k_i)}\right)_{\bar{a},\,t_i b} \tag{16}$$

with $\varepsilon_{ij}=-\varepsilon_{ji}$ being the antisymmetric tensor, $A_{su(k_i)}$ the Cartan matrix of $su(k_i)$ and $\bar{a}=k_i-a$ the antiparticle of a. The insertion of the above phase factor becomes necessary for the same reason as in the simply-laced case [3]. (Note that the charge conjugation in the particle index is missing there). Having taken the square root of the CDD-factor in (15) it still satisfies the bootstrap equation (14) with the possible exception of an overall minus sign. It is exactly this possible sign change which is compensated by the above phase factor and enforces the breaking of parity invariance [4, 3]. This can be explicitly checked when expressing the S-matrix elements S^{ij}_{ab} in terms of hyperbolic functions.

Adopting a similar notation as in [3] the following blocks of meromorphic functions will allow for a compact definition of the S-matrices,

$$[x]_{\theta,ij}=e^{\frac{i\pi x\,\varepsilon_{ij}}{k_{ij}}}\left(\frac{\sinh\frac{1}{2}(\theta+i\pi\frac{x-1+B_{ij}}{k_{ij}})\sinh\frac{1}{2}(\theta+i\pi\frac{x+1-B_{ij}}{k_{ij}})}{\sinh\frac{1}{2}(\theta-i\pi\frac{x-1+B_{ij}}{k_{ij}})\sinh\frac{1}{2}(\theta-i\pi\frac{x+1-B_{ij}}{k_{ij}})}\right)^{\frac{1}{2}} \tag{17}$$

with ε_{ij} being the anti-symmetric tensor already defined above, $t_{ij}=\max(t_i,t_j), k_{ij}=t_{ij}k$ and $B_{ij}=I_{ij}t_j/t_{ij}$ the symmetrized incidence matrix $I=2-A$. This block can be easily seen to have the obvious properties

$$[x]_{\theta,ij}\,[x]_{-\theta,ji}=1 \qquad \text{and} \qquad [k_{ij}-x]_{\theta,ij}=[x]_{i\pi-\theta,ji} \quad \text{for} \quad B_{ij}=1. \tag{18}$$

In the second equality it is implied that one first takes the square root and thereafter performs the shifts in the arguments. Note further that the order of the colour values is relevant, indicating the absence of parity invariance. We are now prepared to define the S-matrix in a closed formula including the two special cases of the same (10) and different colour values (15) discussed before. In terms of meromorphic functions it reads[‡],

$$S^{ij}_{ab}(\theta)=\prod_{\substack{t_{ij}|a/t_i-b/t_j|+1 \\ \text{step}2}}^{t_{ij}(a/t_i+b/t_j)-1}[x]^{A_{ij}t_j/t_{ij}}_{\theta,ij}\quad . \tag{19}$$

[‡]The expression (19) in terms of blocks of meromorphic functions can also be written by use of Coxeter geometry similar as in [3]. However, since we are dealing here only with $su(k_i)$ ATFT (19) is sufficient to check the bootstrap properties.

Besides (19) there is another representation of the S-matrix which is usually of advantage when discussing the thermodynamic Bethe ansatz or the form factor approach to correlation functions [19],

$$S_{ab}^{ij}(\theta) = \eta_{ab}^{ij} \exp \int \frac{dt}{t}\, e^{-it\theta} \left(2\cosh\frac{\pi t}{k_i}\,\delta_{ij} - I_{ij}t_j/t_{ij}\right)\left(2\cosh\frac{\pi t}{k_{ij}} - I_{su(k_{ij})}\right)^{-1}_{\frac{t_{ij}}{t_i}a,\,\frac{t_{ij}}{t_j}b}\,. \tag{20}$$

with the phase factor in front equal to

$$\eta_{ab}^{ij} := \exp i\pi\varepsilon_{ij}\left(A^{-1}_{su(k_{ij})}\right)_{\frac{t_{ij}}{t_i}\bar{a},\,\frac{t_{ij}}{t_j}b}\,. \tag{21}$$

By a straightforward calculation similar to those in the simply-laced case, one can now verify by exploiting (18) that the above S-matrix satisfies the correct bootstrap properties [3, 24]. The complications which arise due to the different fusing structures for different colour values have been already discussed above and are taken into account by a suitable identification of the particles in the $su(k_i)$ and $su(k_j)$ affine Toda model.

Note that the above expressions reduce to the already derived S-matrix of simply-laced HSG models by setting $t_{ij} = t_i = 1$ in accordance with the above definitions. For instance, formula (20) then simplifies to

$$S_{ab}^{ij}(\theta) = e^{i\pi\varepsilon_{ij}[A^{-1}_{su(k)}]_{\bar{a}b}} \exp\int \frac{dt}{t}\, e^{-it\theta}\left(2\cosh\frac{\pi t}{k} - I\right)_{ij}\left(2\cosh\frac{\pi t}{k} - I_{su(k)}\right)^{-1}_{ab}$$

which coincides with earlier expressions found in [4, 17, 3].

2.1 Example $g = B_2$ or $so(5)$

In order to demonstrate the working of the general formulas this section presents a specific example for the choices $g = B_2$ and $k = 3$. The convention is chosen to label by $i = 1$ the long root and by $i = 2$ the short root, i.e. $t_1 = 1, t_2 = 2$. Thus, two copies of the $su(k_1 = 3)$ and $su(k_2 = 6)$ affine Toda model will be coupled to each other. The $su(3)$ copy contains two particle types whose scattering matrix is given by

$$S_{ab}^{11}(\theta) = \begin{pmatrix} [1]^2_{\theta,11} & [2]^2_{\theta,11} \\ [2]^2_{\theta,11} & [1]^2_{\theta,11} \end{pmatrix}. \tag{22}$$

Instead the $su(6)$ copy contains five particles whose scattering amplitudes (19) in terms of the meromorphic blocks (17) read

$$S_{ab}^{22}(\theta) = \begin{pmatrix} [1]^2_{\theta,22} & [2]^2_{\theta,22} & [3]^2_{\theta,22} & [4]^2_{\theta,22} & [5]^2_{\theta,22} \\ [2]^2_{\theta,22} & [1]^2_{\theta,22}[3]^2_{\theta,22} & [2]^2_{\theta,22}[4]^2_{\theta,22} & [3]^2_{\theta,22}[5]^2_{\theta,22} & [4]^2_{\theta,22} \\ [3]^2_{\theta,22} & [2]^2_{\theta,22}[4]^2_{\theta,22} & [1]^2_{\theta,22}[3]^2_{\theta,22}[5]^2_{\theta,22} & [2]^2_{\theta,22}[4]^2_{\theta,22} & [3]^2_{\theta,22} \\ [4]^2_{\theta,22} & [3]^2_{\theta,22}[5]^2_{\theta,22} & [2]^2_{\theta,22}[4]^2_{\theta,22} & [1]^2_{\theta,22}[3]^2_{\theta,22} & [2]^2_{\theta,22} \\ [5]^2_{\theta,22} & [4]^2_{\theta,22} & [3]^2_{\theta,22} & [2]^2_{\theta,22} & [1]^2_{\theta,22} \end{pmatrix}.$$

Note that all blocks appear squared compensating the square root in the definition (17), whence all elements are meromorphic. Identifying the particle $a = 1, 2$ in the $su(3)$ copy with the particles $a = 2, 4$ in the $su(6)$ copy yields then the S-matrix elements

$$S^{12}_{ab}(\theta) = S^{21}_{\bar{b}a}(i\pi - \theta) = \begin{pmatrix} [2]^{-1}_{\theta,12} & [4]^{-1}_{\theta,12} \\ [1]^{-1}_{\theta,12}[3]^{-1}_{\theta,12} & [3]^{-1}_{\theta,12}[5]^{-1}_{\theta,12} \\ [2]^{-1}_{\theta,12}[4]^{-1}_{\theta,12} & [2]^{-1}_{\theta,12}[4]^{-1}_{\theta,12} \\ [3]^{-1}_{\theta,12}[5]^{-1}_{\theta,12} & [1]^{-1}_{\theta,12}[3]^{-1}_{\theta,12} \\ [4]^{-1}_{\theta,12} & [2]^{-1}_{\theta,12} \end{pmatrix} .$$

Again the S matrix elements contain only meromorphic functions, since here the colours are different, whence $B_{12} = B_{21} = 1$ in (17). Note that there now appear additional phase factors in the blocks (17) which break parity invariance. By a straightforward calculation one now verifies the bootstrap equation (3). For example, the one associated to the fusing process $(1, 1) + (1, 1) \rightarrow (2, 1)$ reads

$$S^{12}_{1b}(\theta)S^{12}_{1b}(\theta + i2\pi/3)S^{12}_{1b}(\theta + i4\pi/3) = 1 .$$

3 Conclusions

A new set of solutions to the bootstrap equations has been constructed. Their definition has been motivated by the semi-classical particle spectrum of non simply-laced HSG models. The entirely new feature is the colour structure linked to non simply-laced Lie algebras making it necessary to combine different affine Toda models in one theory. Emphasis has been given to show how these models can be consistently coupled to each other by matching their different fusing structures. The universal formula obtained for the scattering amplitude was observed to contain also the S-matrix of simply-laced HSG models constructed earlier.

The last observation together with the chosen starting point, the semi-classical particle spectrum of HSG theories [6], seem to suggest that (19) might be related to the non simply-laced HSG models, i.e. the integrable perturbation $\Delta = \bar{\Delta} = h^\vee/(k + h^\vee)$ of the coset theory $g_k/u(1)^{\times \operatorname{rank} g}$ [5]. To draw a definite conclusion one would need to identify the conformal field theory to which the proposed S-matrices lead in the ultraviolet limit. As pointed out in the introduction this can be done by means of the thermodynamic Bethe ansatz or a form factor analysis.

In case of the TBA suppose that the solutions of the TBA equations approach a constant value at high energies. One is then in the position to proceed analytically by solving the constant TBA equations along the lines of [16]. The effective central charge c_{eff} in the UV limit would then be given by

$$c_{\text{eff}} = \frac{6}{\pi^2} \sum_{i=1}^{\operatorname{rank} g} \sum_{a=1}^{k_i - 1} \mathcal{L}\left(\frac{x^i_a}{1 + x^i_a}\right) \quad \text{with} \quad x^i_a = \prod_{j=1}^{\operatorname{rank} g} \prod_{b=1}^{k_j - 1} (1 + x^j_b)^{N^{ij}_{ab}} \tag{23}$$

and the exponents read

$$N_{ab}^{ij} = -i \int d\theta \, \frac{d}{d\theta} \ln S_{ab}^{ij}(\theta) = \delta_{ij}\delta_{ab} - A_{ij} t_j / t_{ij} \left(A^{-1}_{su(k_{ij})} \right)_{\frac{t_{ij}}{t_i}a, \frac{t_{ij}}{t_j}b} \quad . \tag{24}$$

Here $\mathcal{L}(x) = \sum_{n=1}^{\infty} x^n/n^2 + \ln x \ln(1-x)/2$ denotes Rogers dilogarithm [25]. This in fact would lead to a discussion similar to the one in [26, 27] for RSOS models giving the correct coset central charge of the non simply-laced HSG models. However, to support that the constant TBA discussion is applicable here further analysis is needed.

Alternatively, one might turn to the form factor approach. For example, all n-particle form factors of the simply-laced $su(N)$ HSG models at level $k = 2$ have been calculated in terms of universal determinant formulas [20, 21]. In the ultraviolet limit this allows to determine the conformal operator spectrum including in particular the perturbing operator $\Delta = \bar{\Delta} = h^\vee/(k + h^\vee)$ [21]. In this manner the S-matrices constructed in [4] could be unambiguously related to the semi-classical discussion in [5]. One may now extend this analysis also to the non simply-laced case with the integral formula (20) yielding the starting point, since from it the minimal two-particle form factor can be immediately constructed. In contrast to the simply-laced HSG models considered so far the non simply-laced theories proposed here will turn out to be more complicated, since already at level $k = 2$ one has to deal with bound state poles, a feature which is absent for simply-laced Lie algebras.

Acknowledgments: I would like to thank Andreas Fring and J.Luis Miramontes for careful reading of the manuscript and valuable comments. This work has been started at the Freie Universität Berlin and has been finished at the YITP. The financial support of the Deutsche Forschungsgemeinschaft (Sfb288) and the National Science Foundation (Grants DMR0073058 and PHY9605226) is gratefully acknowledged.

References

[1] B. Schroer, T.T. Truong and P. Weisz, *Phys. Lett.* **B63** (1976) 422;
M. Karowski, H.J. Thun, T.T. Truong and P. Weisz, *Phys. Lett.* **B67** (1977) 321;
A.B. Zamolodchikov, *JETP Lett.* **25** (1977) 468.

[2] C.N. Yang, *Phys. Rev. Lett.* **19** (1967) 1312;
R.J. Baxter, *Ann. Phys. (NY)* **70** (1972) 323.

[3] A. Fring and C. Korff, *Phys. Lett.* **B477** (2000) 380.

[4] C.R. Fernández-Pousa and J.L. Miramontes, *Phys. Lett.* **B472** (2000) 392.

[5] C.R. Fernández-Pousa, M.V. Gallas, T.J. Hollowood and J.L. Miramontes, *Nucl. Phys.* **B484** (1997) 609;*Nucl. Phys.* **B499** (1997) 673.

[6] C.R. Fernández-Pousa and J.L. Miramontes, *Nucl. Phys.* **B518** (1998) 745.

[7] A.V. Mikhailov, M.A. Olshanetsky and A.M. Perelomov, *Commun. Math. Phys.* **79** (1981) 473;

G. Wilson, *Ergod. Th. Dyn. Syst.* **1** (1981) 361;

D.I. Olive and N. Turok, *Nucl. Phys.* **B257** [FS14] (1985) 277.

[8] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, *Phys. Lett.* **B87** (1979) 389;

H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Nucl. Phys.* **B338** (1990) 689;

C. Destri and H.J. De Vega, *Phys. Lett.* **B233** (1989) 336, *Nucl. Phys.* **B358** (1991) 251;

P.G.O. Freund, T.R. Klassen and E. Melzer, *Phys. Lett.* **B229** (1989) 243;

P. Christe and G. Mussardo, *Nucl. Phys.* **B330** (1990) 465, *Int. J. Mod. Phys.* **A5** (1990) 4581.

G. Mussardo, *Phys. Rep.* **218** (1993) 215.

[9] L. Castillejo, R.H. Dalitz and F.J. Dyson, *Phys.Rev.* **101** (1956) 453.

[10] R. Köberle and J.A. Swieca, *Phys. Lett.* **B86** (1979) 209;

A.B. Zamolodchikov, *Int. J. Mod. Phys.* **A1** (1989) 4235.

[11] V.A. Fateev and A.B. Zamolodchikov, *Sov. Phys. JETP* **62** (1985) 215.

[12] D. Gepner, *Nucl. Phys.* **B290** (1987) 10;

D. Gepner and Z. Qiu, *Nucl. Phys.* **B285** (1987) 423.

[13] P. Goddard, A. Kent and D.Olive, *Phys. Lett.* **B152** (1985) 88;

[14] E. Witten, *Commun. Math. Phys.* **92** (1984) 455;

[15] C.N. Yang and C.P. Yang, *Phys. Rev.* **147** (1966) 303; *J. Math. Phys.* **10** (1969) 1115.

[16] Al.B. Zamolodchikov, *Nucl. Phys.* **B342** (1990) 695;

T.R. Klassen and E. Melzer, *Nucl. Phys.* **B338** (1990) 485; **B350** (1991) 635.

[17] O.A. Castro-Alvaredo, A. Fring, C. Korff and J.L. Miramontes, *Nucl. Phys.* **B573** (2000) 535.

[18] G. Dunne, I. Halliday and P. Suranyi, *Nucl. Phys.* **B325** (1989) 526.

[19] P. Weisz, *Phys. Lett.* **B67** (1977) 179;

M. Karowski and P. Weisz, *Nucl. Phys.* **B139** (1978) 445;

F.A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, World Scientific, Singapore, (1992).

[20] O.A. Castro-Alvaredo, A. Fring and C. Korff, *Phys. Lett.* **B484** (2000) 167.

[21] O.A. Castro-Alvaredo and A. Fring, *Identifying the operator content, the Homogeneous Sine-Gordon models*, hep-th/0008044; *Renormalization group flow with unstable particles*, hep-th/0008208; *Decoupling the SU(N)$_2$ homogeneous Sine-Gordon model*, hep-th/0010262

[22] G.W. Delius, M.T. Grisaru and D. Zanon, *Nucl. Phys.* **B382** (1992) 365.

[23] A. Fring, C. Korff and B.J. Schulz, *Nucl. Phys.* **B567** (2000) 409.

[24] C. Korff, *Lie algebraic structures in integrable models, affine Toda field theory*, dctoral thesis, July 2000, Freie Universität Berlin; http://darwin.inf.fu-berlin.de/2000/99/

[25] L. Lewin, Dilogarithms and associated functions (Macdonald, London, 1958).

[26] A.N. Kirillov and N.Y. Reshetikhin, *J. Sov. Math.* **52** (1990) 3156.

[27] A. Kuniba, *Nucl. Phys.* **B389** (1993) 209;

A. Kuniba and T. Nakanishi, *Mod. Phys.* **7** (1992) 3487;

A. Kuniba, T. Nakanishi and J. Suzuki, *Mod. Phys.* **8** (1993) 1649; *Int. J. Mod. Phys.* **9** (1994) 5215; *Int. J. Mod. Phys.* **9** (1994) 5267.